Derivation of the tachyonic Dirac-type wave equation

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March 3, 2025

We start from the energy-momentum vector expression $\vec{P}$

$$\vec{P}c = E\vec{e} + \vec{p}c \tag{1}$$

We consider all vectors to be 4-dimensional, those in 3-dimensional space having the time component zero.

$$\vec{p} = 0\,\vec{e} + p_x\vec{i} + p_y\vec{j} + p_z\vec{k} \tag{2}$$

where

$$\begin{pmatrix} \vec{e} \\ \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix} \cdot \begin{pmatrix} \vec{e} & \vec{i} & \vec{j} & \vec{k} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta \tag{3}$$

$$\vec{p}^2 = \vec{p}\cdot\vec{p} = 0\,\vec{e}^2 + p_x^2\,\vec{i}^2 + p_y^2\,\vec{j}^2 + p_z^2\,\vec{k}^2 = -\left(p_x^2 + p_y^2 + p_z^2\right) = -p^2 \tag{4}$$

$$p = \sqrt{-\vec{p}^2} \tag{5}$$

$$\vec{P}^{\,2} = P^T \cdot \eta \cdot P => P = \sqrt{-\vec{P}^{\,2}} \tag{6}$$

We further distinguish two cases:
1)
Dirac equation, valid for tardyonic waves (speed v less than the speed of light c).

$$\frac{pc}{E} = \frac{v}{c} < 1 \tag{7}$$

$$\frac{E^2}{c^2} - p^{\,2} = m_0^2c^2 \quad => \quad (\pm E)^2 = (\pm pc)^2 + (\pm m_0c^2)^2 \tag{8}$$

m_0 is the rest mass and P is the momentum at infinite speed.
Applying the quantization rule from elementary quantum mechanics:

$$E \to i\hbar\frac{\partial}{\partial t}\ , \quad \vec{p} \to \frac{\hbar}{i}\vec{\nabla} \tag{9}$$

we obtain from equation (8)

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2 \psi_\sigma = \left(\frac{\hbar c}{i}\vec{\nabla}\right)^2 \psi_\sigma + m_0^2c^4\psi_\sigma \tag{10}$$

This is the Klein-Gordon equation for each σ component of the ψ spinor wave function. We look for a linear equation for the wave function ψ

$$\pm i\hbar\frac{\partial\psi}{\partial t} = \pm\frac{\hbar c}{i}\left(\alpha^j\frac{\partial\psi}{\partial x^j}\right) \pm m_0c^2\beta\psi \tag{11}$$

This is a first form of the well-known Dirac equation.

2)

Dirac-type equation for tachyon waves (speed v greater than the speed of light c)

$$\frac{pc}{E} = \frac{v}{c} > 1 \tag{12}$$

$$\vec{p}^2 - \frac{E^2}{c^2} = P^2 \qquad => \qquad E^2 = (pc)^2 - P^2c^2 \tag{13}$$

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar c}{i}\left(\alpha^j\frac{\partial\psi}{\partial x^j}\right) \pm iPc\beta\psi \tag{14}$$

This is the Dirac-type equation for tachyon waves.

$$\begin{cases} \alpha^k\alpha^j + \alpha^j\alpha^k = 2\delta^{kj}\mathbb{1}_4 & (15) \\ \alpha^k\beta + \beta\alpha^k = 0 & (16) \\ \beta^2 = \mathbb{1}_4 & (17) \\ \alpha^{k\,\dagger} = \alpha^k & (18) \\ \beta^\dagger = \beta & (19) \end{cases}$$

$$\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} \tag{20}$$

$$\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \tag{21}$$

$$\mathbb{1}_4 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \tag{22}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{23}$$