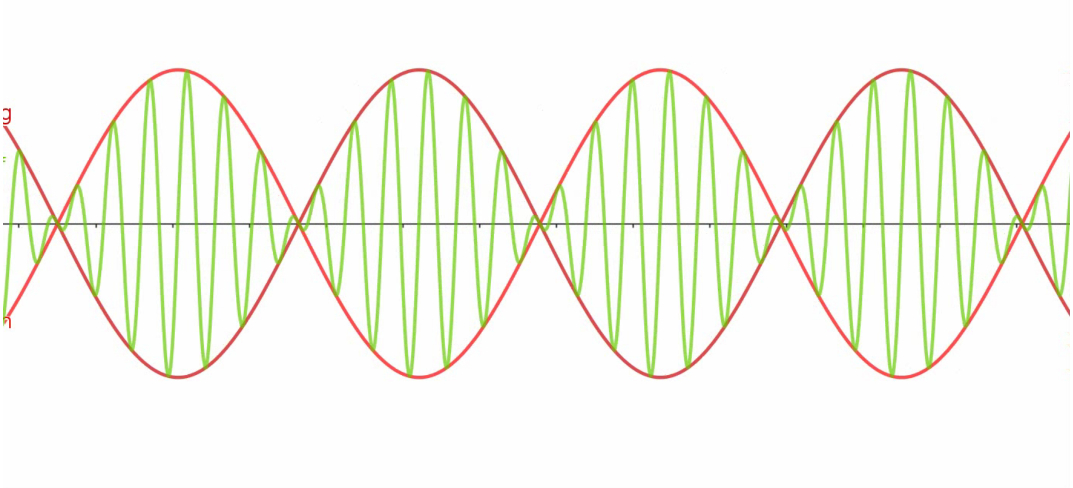

In this paper we use the following hyperbolic identities:

$$\begin{cases} \sinh x = \dfrac{\tanh x}{\sqrt{1-\tanh^2 x}} = \dfrac{1}{\sqrt{\coth^2 x - 1}} & (1) \\ \cosh x = \dfrac{1}{\sqrt{1-\tanh^2 x}} = \dfrac{\coth x}{\sqrt{\coth^2 x - 1}} & (2) \end{cases}$$

The propagation of a relativistic (electromagnetic) plane wave in relativistic spacetime time along the x-axis and of amplitude $A = 1$ is described by the following complex function:

$$e^{i(\omega t - kx)} \tag{3}$$

The position vector in hyperbolic spacetime is:

$$\vec{s} = ct\,\vec{e} + x\,\vec{i} \tag{4}$$

where the versors of the $\vec{e}$ and $\vec{i}$ axes have the properties:

$$\vec{e}\cdot\vec{i} = 0; \quad \vec{e}^{\,2} = 1; \quad \vec{i}^{\,2} = -1 \tag{5}$$

We write the wave phase successively:

$$\omega t - kx = \left(\frac{\omega}{c}\vec{e} + k\vec{i}\right)\cdot(ct\,\vec{e} + x\,\vec{i}) \tag{6}$$

To simplify further calculations we make the following notations:

$$\vec{\Omega} = \frac{\omega}{c}\vec{e} + k\vec{i} \tag{7}$$

We continue the math.
1)
If $\dfrac{\omega}{c} > k$ then:

$$\Omega^2 = \frac{\omega^2}{c^2} - k^2 \tag{8}$$

There is a unique α with the properties:

$$\begin{cases} \omega = c\Omega\cosh\alpha = \Omega\dfrac{v_\varphi}{\sqrt{\dfrac{v_\varphi^2}{c^2} - 1}} & (9) \\ k = \Omega\sinh\alpha = \Omega\dfrac{1}{\sqrt{\dfrac{v_\varphi^2}{c^2} - 1}} & (10) \end{cases}$$

$$v_\varphi = \frac{\omega}{k} = c\coth\alpha \tag{11}$$

$$\begin{cases} d\omega = c\Omega \sinh \alpha d\alpha & (12) \\ dk = \Omega \cosh \alpha d\alpha & (13) \end{cases}$$

$$v_g = \frac{d\omega}{dk} = \frac{c^2}{v_\varphi} \tag{14}$$

2)
If $\frac{\omega}{c} < k$ then:

$$\Psi^2 = k^2 - \frac{\omega^2}{c^2} = -\Omega^2 \tag{15}$$

$$\begin{cases} \omega = c\Psi \sinh \alpha = \Psi \frac{v_\varphi}{\sqrt{1 - \frac{v_\varphi^2}{c^2}}} & (16) \\ k = \Psi \cosh \alpha = \frac{\Psi}{\sqrt{1 - \frac{v_\varphi^2}{c^2}}} & (17) \end{cases}$$

$$\frac{\omega}{k} = c \tanh \alpha = v_\varphi \tag{18}$$

$$\begin{cases} d\omega = c\Psi \cosh \alpha d\alpha & (19) \\ dk = \Psi \sinh \alpha d\alpha & (20) \end{cases}$$

$$\frac{d\omega}{dk} = c \coth \alpha = v_g > c \tag{21}$$

In both cases:

$$\boxed{v_\varphi v_g = c^2} \tag{22}$$

Consider the superposition of two electromagnetic waves of different but close frequencies.

$$\psi = e^{i\vec{\Omega}_1 \cdot \vec{s}} + e^{i\vec{\Omega}_2 \cdot \vec{s}} = e^{i\vec{\Omega} \cdot \vec{s}}(e^{i(\Delta\vec{\Omega} \cdot \vec{s})} + e^{-i(\Delta\vec{\Omega} \cdot \vec{s})}) \tag{23}$$

where

$$\begin{cases} \vec{\Omega} + \Delta\vec{\Omega} = \vec{\Omega}_1 & (24) \\ \vec{\Omega} - \Delta\vec{\Omega} = \vec{\Omega}_2 & (25) \end{cases}$$

$$\vec{\Omega} = \frac{\vec{\Omega}_1 + \vec{\Omega}_2}{2} \tag{26}$$

$$\Delta\vec{\Omega} = \frac{\vec{\Omega}_1 - \vec{\Omega}_2}{2} \tag{27}$$

$$\vec{\Omega} \cdot \vec{s} = \omega t - kx \tag{28}$$

$$\Delta\vec{\Omega} \cdot \vec{s} = \Delta\omega t - \Delta kx \tag{29}$$

Plane wave equation

$$\psi = 2e^{ik(v_\varphi t - x)} \cos \Delta k(v_g t - x) \tag{30}$$

Subluminal group velocity: please take a look at the animation here
Superluminal group velocity: please take a look at the animation here
Light velocity: please take a look at the animation here