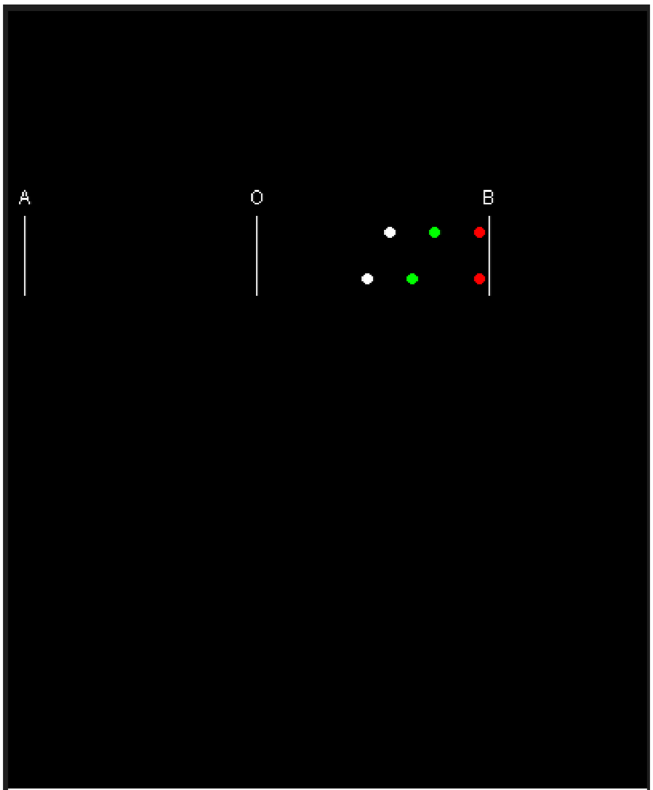
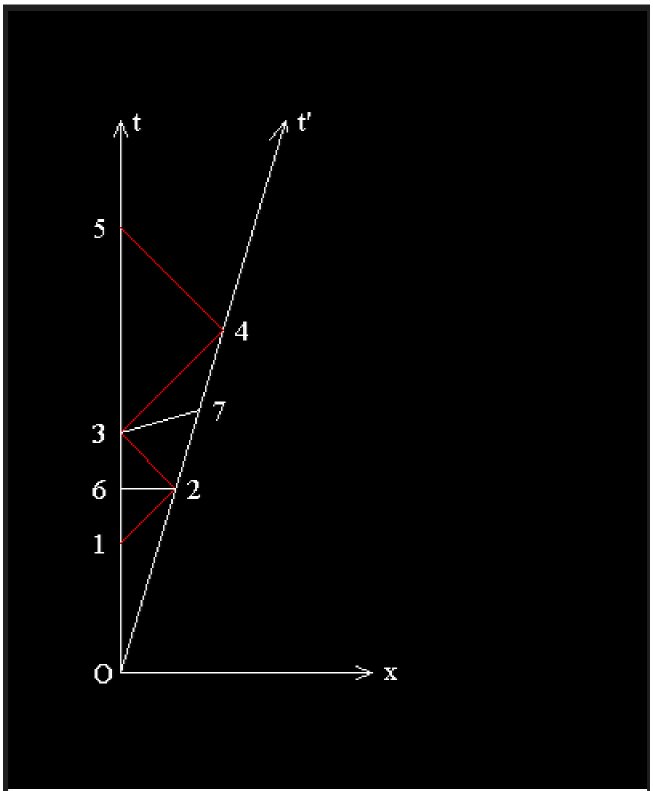
Minkowski Geometry

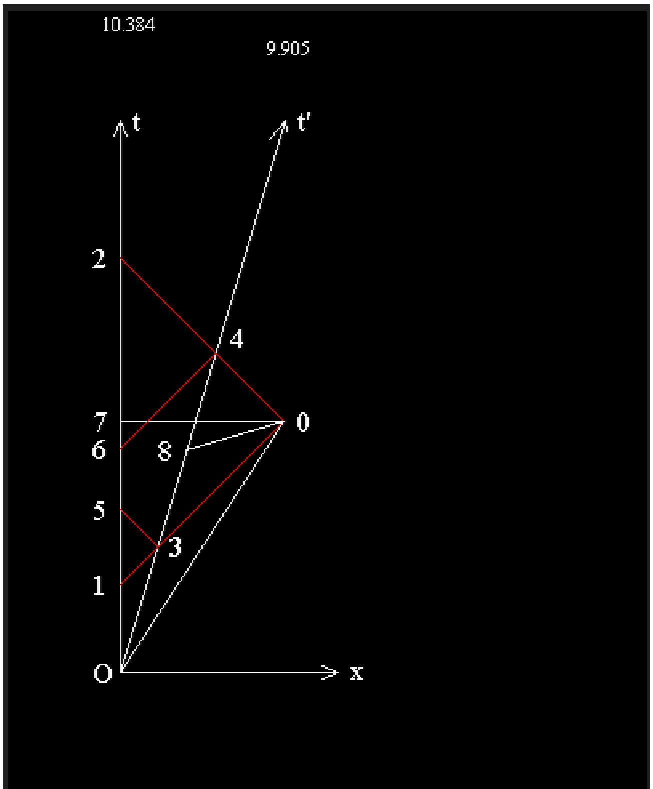
December 8, 2022



Animation 1.



Animation 2.



Animation 3.

The measurement of time is based on periodic processes. A periodic process is a process that occurs in continuously repeating identical segments. Such

a process is useful as a more or less accurate clock. A certain number of repetitions of such process sections defines the unit of time. Particularly regular periodic processes are pendulum oscillations, elastic oscillations, atomic oscillations and the rotation of the earth.
In classical physics a clock once calibrated will theoretically always show the correct time, no matter how it moves and everywhere in space. Two clocks running synchronously side by side and at relative standstill are sent on an arbitrary journey through space. After that they are brought side by side again. Also now the clocks run synchronously. This is the postulate of absolute time in classical physics.
We want to represent movements graphically. A point in motion through space is determined if its position is known at any time. To specify the position and the time, we need a reference frame. A reference system consists of a coordinate system with a clock at the origin and an observer who can process all the data. The movement is concretely indicated by the three coordinates as functions of the time. If x, y and z are the Cartesian coordinates and t is the time, then the motion is completely described by the functions:

$$x = x\left(t\right) \tag{1}$$

$$y = y\left(t\right) \tag{2}$$

$$z = z\left(t\right) \tag{3}$$

We want to represent these functions graphically. Of course, this is only possible to a limited extent, because we cannot represent more than three dimensions. We would need four dimensions, three for space and one for time. But we can only represent one or two dimensions of space and, as the third dimension of the graph, time. The graphic thus created, we will call space-time diagram in the following. This is still nothing more than the graphic representation of the movement, no ”space-time” and has still no independent physical reality.
A point in the space-time diagram is called an event. The graphic representation of the movement of a point is called world line of the point. The motion of a point with constant velocity is represented by a straight line.
And now we try to define a norm in the space-time diagram. To do this, we give the space-time diagram the structure of a vector space. We write a vector **x** in the following way:

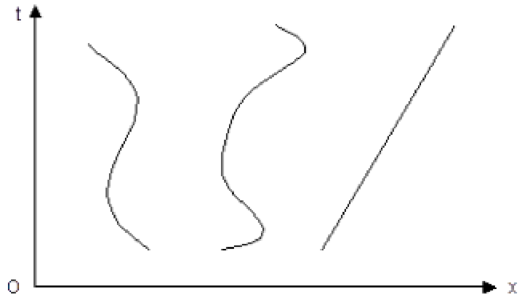


Figure 1:

$$\vec{x} = t\vec{E} + x\vec{I} \tag{4}$$

$\vec{E}$ is any vector on the t-axis and $\vec{I}$ on the x-axis.
If $|\vec{x}|$ denotes a norm from the vector $\vec{x}$ and λ is any real scalar, then the definition axiom of the norm states:

$$|\lambda\vec{x}| = |\lambda| \cdot |\vec{x}| \tag{5}$$

If two norms are defined, then both satisfy condition (2.) and therefore condition:

$$\frac{|\lambda\vec{x}|_1}{|\vec{x}|_1} = \frac{|\lambda\vec{x}|_2}{|\vec{x}|_2} \tag{6}$$

We define a norm such that we associate with each vector $\vec{x}$ a unit vector $\vec{x}^0$ in its direction, i.e.

$$\vec{x} = \lambda\vec{x}^0 \tag{7}$$

and which has the same norm in all directions (see Figure 2).If the vector $\vec{x}^0$ takes all directions around the origin of the coordinate system, it describes a three-dimensional hypersurface in the space-time diagram. All different norms that can be defined differ by the shape of this hypersurface.
The hypersurface defined in this way we call unit surface. So, to define the norm associated to this unit surface, we intersect the straight line defined by the vector $\vec{x}$ with the unit surface, obtaining the vector, and the norm is defined using another arbitrary norm (e.g., the Euclidean) of the following dimensions:

$$|\vec{x}| = \frac{|\vec{x}|_{Euklidisch}}{|\vec{x}^0|_{Euklidisch}} \cdot |\vec{x}^0| \tag{8}$$

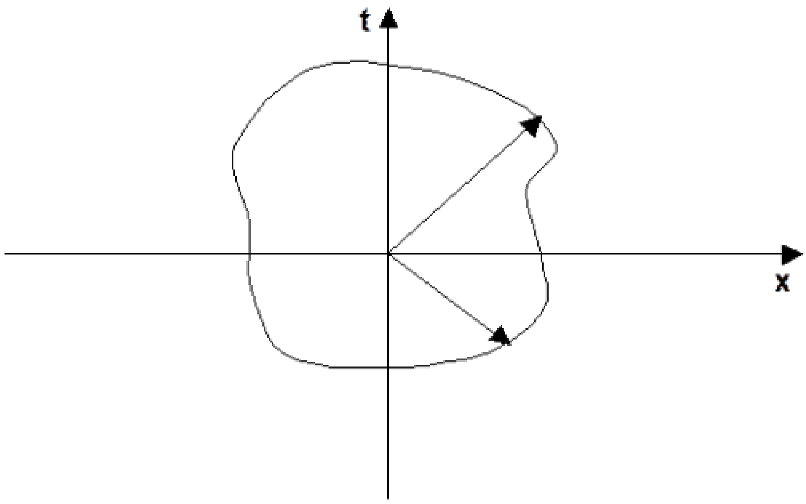


Figure 2:

Let us apply these findings to a classical space-time diagram. Let us define the unit surface as a hyperplane parallel to space (see Figure 2.). The constant value for $|\vec{x}^0|$ we choose:

$$\left|\vec{x}^{\,0}\right| = 1 \tag{9}$$

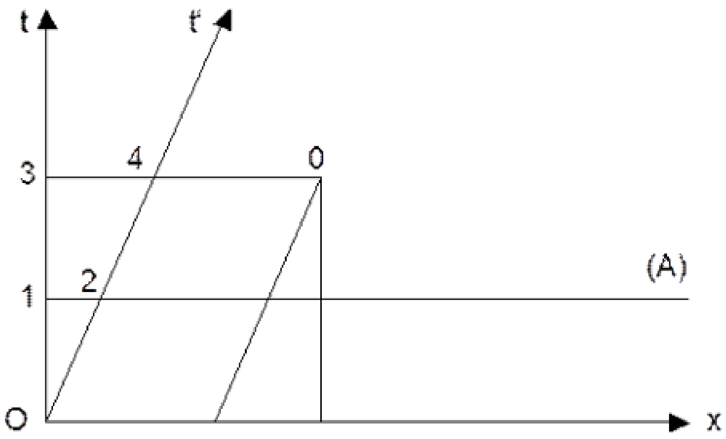


Figure 3:

In the following, we will use the points with numbers: 1, 5, 13,... or with letters: A, B, O,... etc. with a colon as separator. The straight lines with $\{10 : 12\}$ where 10 and 12 are two points. Vectors with $|10 : 12\rangle$, lines $[10 : 12]$, norms $|10 : 12|$. We want to represent the Galilean transformation geometrically. One projects the point 0 on the coordinate axes and one determines the distances of the axes along between projection and coordinate origin with the help of the norm. The meaning of different geometrical objects is the following.

$$\vec{x}^0 = |O:1\rangle\,, \quad \vec{x}'^0 = |O:2| \quad x = |3:0|\,, \quad x' = |4:0|\,, \quad vt = |3:4| \tag{10}$$

The norm is any norm satisfying axiom (5). The unit surface in our 2-dimensional space-time diagram is the straight line $\{1:(A)\}$. Because $\{1:2\}$ is parallel to $\{3:4\}$, we have:

$$\frac{|O:3|}{|O:1|} = \frac{|O:4|}{|O:2|} \tag{11}$$

By defining the times, we have

$$t = \frac{|O:3|}{|O:1|}, \quad t' = \frac{|O:4|}{|O:2|}, \quad t' = t \tag{12}$$

From figure 3 it can be seen:

$$|4:0| = |3:0| - |3:4| \tag{13}$$

$$x' = x - vt \tag{14}$$

But this is the Galilean transformation. The space-time diagram is not in contradiction with Galilean physics. We will show that it is not in contradiction with Einsteinian physics either. Everything points to the fact that the processes take place in a four-dimensional physically existing space-time, whose one possible representation is the space-time diagram. Having recorded the movements, we examine these records. And not in original size, but reduced so that it fits on a piece of paper or on this screen. Especially the lengths and the velocities are drastically reduced. Even if no velocities greater than that of light existed in the universe, on our model the speed of light c is reduced to, say, 1cm/s, and we can use true light to observe the whole thing. We then examine the recorded past, and we can run it at a speed greater than c, and move backwards in time without any contradiction. Something just happened or didn't happen, we can't change anything in the record without destroying the association between past reality and record. If we represent the movements graphically, we make nevertheless the step to consider the time as a coordinate. A serious consequence of this is that the absoluteness of the simultaneity of two events occurring at different places in space must be abandoned. A central issue is the propagation of electromagnetic waves, including light. In accordance with the experimental results, we postulate that the light propagates in the same way, regardless of the movement of the light source. So, the light is released into the ether, which then takes care of the propagation, without the initial velocity playing a role. This is not the case for balls or grant balls, their motion depends on the initial velocity with which they were shot. Actually, we cannot speak of velocity yet, because we have not yet defined simultaneity and the measurement of time. Therefore, we cannot speak of the speed of light at all yet. We can illustrate this postulate in the animation 1. (Include animation) The two white points correspond to, say, two spaceships moving at different speeds. They meet at point O, from where they emit two light signals (the red dots), together with two projectiles (the green dots) from identical firing devices. The light signals reach target B together, but the spaceships and the projectiles separately. We will establish this property of light to propagate independently of the speed of its source as an axiom of relativity. Be careful, this is not the constancy of the speed of light, i.e. the speed of the red points with respect to the white ones. By the way, this is also true, but needs the proof. Here perhaps only so much: the time measured by identical clocks in the spaceships is not absolute, but depends on the movement of the clocks. About it however something later in this letter.

If we graphically represent the absoluteness of light propagation, it looks like in Figure 4: The observer with worldline $\{1\}$ emits a photon whose motion is described by worldline $\{A\}$. The same for observer $\{2\}$ with world line of photon $\{B\}$. The axiom of the absoluteness of the light propagation says that the straight lines $\{A\}$ and $\{B\}$ are parallel in the space-time diagram. Having created a space-time diagram for the motion of different points with the help of a coordinate system, we will delete the coordinate axes and keep only the world lines. We establish the axiom that the world lines are absolute, regardless of whatever coordinate system we may draw in the space-time diagram. We want to define now the simultaneity as an observer equipped with an arbitrary clock with a straight world line detects it. We assume that the observer sends the picture of his clock embedded in a TV signal constantly as a spherical wave and receives the reflections of the signal. Let any event E be the reflection of this signal. We define the time of the event measured by this clock as the middle M between the emission of the signal A and the reception of its reflection B on the world line of the clock (see

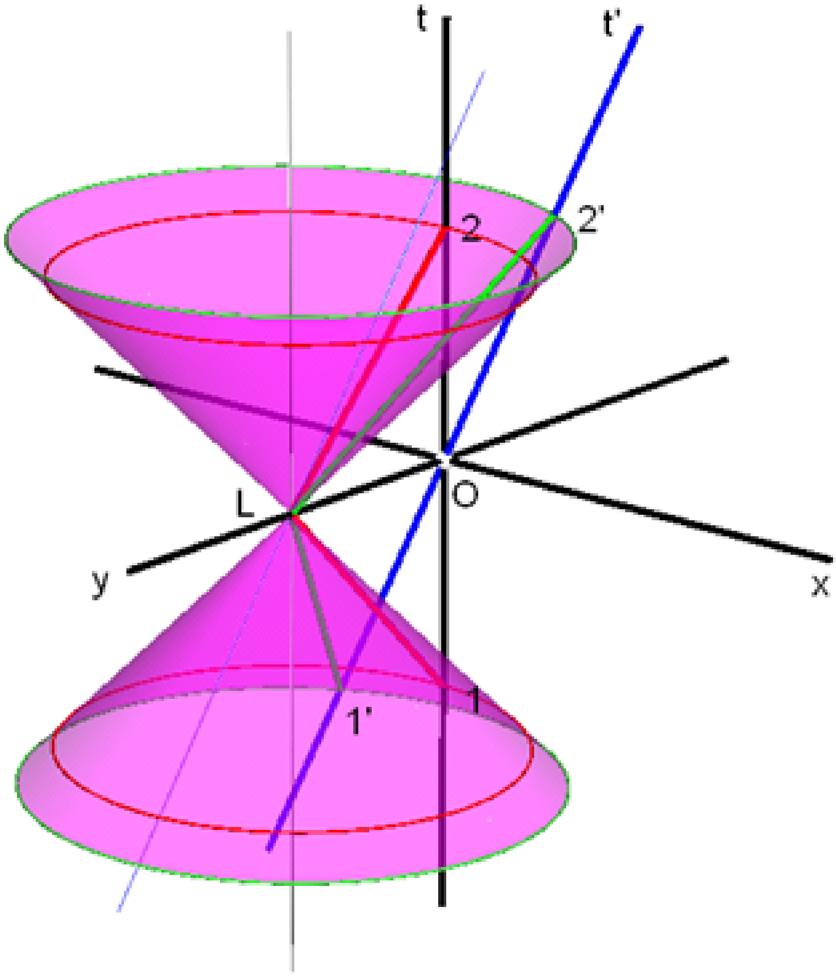


Figure 4:

figure 4). A standard relativistic mental clock was introduced to use the absolute propagation of light for the measurement of time. The back and forth movement of a light pulse between two parallel mirrors serves as a periodic process. Let us now consider two inertial systems that measure time using light pulse clocks.

But we need to take the length of the two light pulse clocks equal, without stopping their movement so that they could be superimposed and compared. Therefore we choose the distance [O:L] along the y axis as the starting position of the two clocks. This is correct only if the events O and L are simultaneous in both reference systems. We now want to prove that this is the case. The observer B emits a spherical light wave (event 1) which is reflected by the mirror of the light pulse clock and is received in event 2. The same is true for observer B'. We have to show that O is the center of both the

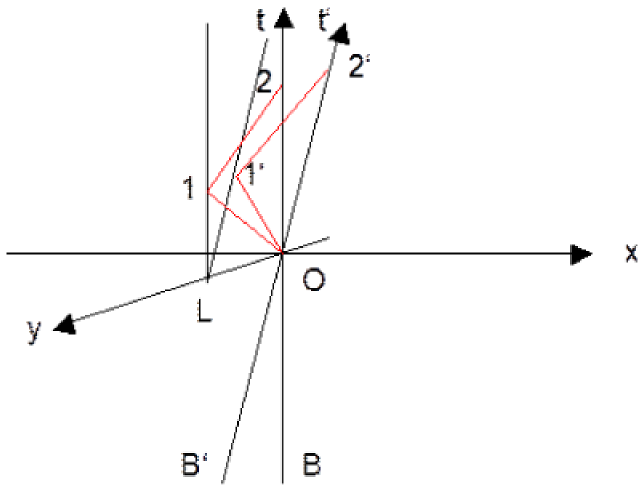


Figure 5:

distance [1:2] and the distance [1':2']. The wavefront emitted from 1 (with z = 0) has the equation:

$$x^2 + y^2 = c^2 (t - t_1)^2 \tag{15}$$

The quantity c is a suitably chosen constant needed to write the equations of various geometric objects in the space-time diagram. After the exact definition of the unit of time and length, it will turn out that c is the speed of light in the coordinate system O:t:x:y. The provisional absence of the units must not bother us further, because we work in a geometrical representation, where all units are lengths anyway. The exact assignment between units in reality and the scaling of the axes, comes later. Therefore we do not work with numerical values, but with identifiers. Event L (x=0, y= l, t=0) belongs to this wavefront. We put its coordinate in the upper equation and get the equation:

$$l^2 = c^2 {t_1}^2 \tag{16}$$

We solve for t_1 and obtain:

$$t_1 = -\frac{l}{c} \tag{17}$$

The wave reflected in L in all directions has the equation:

$$x^2 + (y - l)^2 = c^2 t^2 \tag{18}$$

This intersects with the $\{O : t\}$ axis (x=0, y=0) in event 2:

$$l^2 = c^2 t_2^2, \quad t_2 = \frac{l}{c} \tag{19}$$

The center between 1 and 2 has the t-coordinate:
par

$$\frac{t_1 + t_2}{2} = \frac{1}{2}\left(-\frac{l}{c} + \frac{l}{c}\right) = 0 \tag{20}$$

The x-coordinate was always 0, everything took place in the plane $\{O : t : y\}$. So, the middle between 1 and 2 is the event O. Similarly, we show that the middle between 1' and 2' is also O. So, L and O are simultaneous events for both observer B and B'. We use the distance [O:L] as the initial position of the two light pulse clocks.

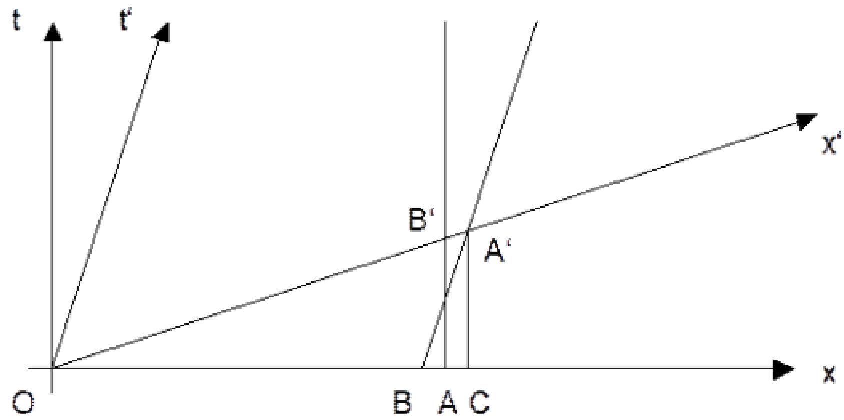


Figure 6:

The axis $\{O : t\}$ is also the worldline of the mirror S_1 of the light pulse clock of observer B. The worldline of the mirror S_2 of the same is $\{L : 1\}$. The same is true for observer B'. In the event O both observers emit a spherical light wave. This wave intersects the world line of mirror S_2 in event 1 and that of mirror S_2' in event 1'. Figure 6 is self-explanatory. The unit time in the $\{O : t\}$ direction is $\{O : 2\}$ and in the $\{O : t'\}$ direction is $\{O : 2'\}$. Let us denote $\{O : L\} = $ l. The spherical wave from O has the equation:

$$x^2 + y^2 = c^2 t^2 \tag{21}$$

This wave intersects with the world line of the mirror S_2 in event 1, and with the world line of the mirror S_2' in 1'. The straight line $\{L : 1\}$ has the equations:

$$x = 0, \quad y = l \tag{22}$$

Substituting (22) into (21), we get the coordinates of the intersection:

$$t_1 = \frac{l}{c}, \quad x_1 = 0, \quad y_1 = l \tag{23}$$

The light wave reflected in event 1 has the equation:

$$(x - x_1)^2 + (y - y_1)^2 = c^2 (t - t_1)^2 \tag{24}$$

This wave again reaches observer B in event 2. And so on, and so on.... We write the results, namely the coordinates of events 2 and 2':

$$t_2 = \frac{2l}{c}, \quad x_2 = 0, \quad y_2 = 0 t_{2'} = \frac{2l/c}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad x_{2'} = v t_{2'}, \quad y_{2'} = 0 \tag{25}$$

If we eliminate from these equations, v and l, we get the equation of unit area:

$$c^2 t_{2'}^2 - x_{2'}^2 = c^2 t_2^2 \tag{26}$$

This is the equation of a hyperbola in the space-time diagram. The vector $\vec{x}^0$ describes a hyperbola in the 2-dimensional space-time diagram. We write this vector as follows:

$$\vec{x}^0 = t_0 \vec{E} + x_0 \vec{I} \tag{27}$$

where $\vec{E}$ is the unit vector afflicted with measure in the direction of the axis $\{O : t\}$ and $\vec{I}$ in the direction of the axis $\{O : x\}$. By t_0 and x_0 we have denoted the previous $t_{2'}$ and $x_{2'}$ when $t_2 = 1$.

$$\vec{x}^0 = \frac{\vec{E} + v\vec{I}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{28}$$

We try to define the scalar product of $\vec{x}^0$ with itself and determine the scalar products between the basis vectors $\vec{E}$ and $\vec{I}$ among themselves. Then it is easy to define the scalar product between any two vectors.

$$(\vec{x}^{\,0})^2 = \frac{\vec{E}^{\,2} + 2v\vec{E}\cdot\vec{I} + v^2\vec{I}^2}{1 - \dfrac{v^2}{c^2}} \tag{29}$$

$$1 - \frac{v^2}{c^2} = \frac{\vec{E}^2}{(\vec{x}^{\,0})^2} + 2v\frac{\vec{E}\cdot\vec{I}}{(x^{\,0})^2} + v^2\frac{\vec{I}^2}{(\vec{x}^{\,0})^2}$$

The last equation must hold for all v's, so:

$$\frac{\vec{E}^2}{(\vec{x}^{\,0})^2} = 1, \quad \frac{\vec{E}\cdot\vec{I}}{(\vec{x}^{\,0})^2} = 0, \quad \frac{\vec{I}^2}{(\vec{x}^{\,0})^2} = -\frac{1}{c^2} \tag{30}$$

Here we can choose a quantity by convention, e.g. $\vec{I}^{\,2} = 1$ as in 3-dimensional space. The other scalar products are then uniquely defined:

$$\left(\vec{x}^{\,0}\right)^2 = -c^2, \quad \vec{E}^{\,2} = -c^2, \quad \vec{E}\cdot\vec{I} = 0 \tag{31}$$

.

We are forced to admit negative scalar products of vectors with themselves. Thus, the norm is not positively defined. The zero scalar product still means orthogonality. If we normalize the basis vectors, we get for the vector $\vec{x}^{\,0}$:

$$\vec{x}^{\,0} = t_0\sqrt{\left|\vec{E}^{\,2}\right|}\frac{\vec{E}}{\sqrt{\left|\vec{E}^{\,2}\right|}} + x_0\sqrt{\left|\vec{I}^{\,2}\right|}\frac{\vec{I}}{\sqrt{\left|\vec{I}^{\,2}\right|}} \tag{32}$$

For any vector $\vec{x}$ we drop the index 0:

$$\vec{x} = ct\frac{\vec{E}}{\sqrt{-\vec{E}^2}} + x\frac{\vec{I}}{\sqrt{\vec{I}^2}} = ct\vec{e} + x\vec{i} \tag{33}$$

The newly introduced basis vectors $\vec{e}$ and $\vec{i}$ have the properties:

$$\vec{e}^{\,2} = \frac{\vec{E}^2}{-\vec{E}^2} = -1, \quad \vec{i}^{\,2} = \frac{\vec{I}^2}{\vec{I}^2} = 1, \quad \vec{e}\cdot\vec{i} = 0 \tag{34}$$

Because this norm is not positively defined, in what follows we will speak of the quadratic norm, which from now on we will simply call norm. Its axiom of definition is:

$$\|\lambda\vec{x}\| = \lambda^2\,\|\vec{x}\| \tag{35}$$

The norm defined by the above scalar product is in the 2-dimensional space-time diagram:

$$\|\vec{x}\| = \vec{x}^{\,2} = -c^2t^2 + x^2$$

(36)

This is called the Minkowskian norm. Einstein's entire special theory of relativity is based on it. We call the first order norm length. It is given by:

$$|\vec{x}| = \sqrt{-\vec{x}^2} = \sqrt{c^2t^2 - x^2} \tag{37}$$

For now, it is defined only for events for which ct is greater than x. Because the norm is not defined positively, it can also be zero for x=ct. Vectors with negative norm (in the sense of formula (36) are called "time-like", with norm zero "light-like" and with positive norm "space-like". But what is the geometrical and physical interpretation of the length of a time-like vector? To answer this question, let us consider animation 2. Two observers with clocks constantly send TV signals with the image of their clocks. Each of them also receives the TV signal of the other. Thus, each observer sees his own clock and the image of the other's clock. The time in the picture runs slower than the time displayed by the clock. The observers are equal exactly when the ratio of the two displays, clock and picture, is equal for both observers. In animation 2, {O:t} is the world line of observer B, and {O:t'} is the world line of observer B'. In event 2, observer B"s clock shows time t_2', and B's clock image shows time t_1. In event 3, observer B's clock displays time t_3, and the image received by B', displays time t_2'. Because of the equality of B and B', we must have:

$$\frac{t_2'}{t_1} = \frac{t_3}{t_2'} \tag{38}$$

thus

$$t_2' = \sqrt{t_1 t_3} \tag{39}$$

Let's calculate the coordinates of points 1, 2, and 3. We assume the coordinates of point 2 to be known (t_2, x_2). The straight line $\{1 : 2\}$ has slope c and contains the point 2, so it has the equation: $x - x_2 = c\,(t - t_2)$ with $x_2 = vt_2$ This straight line intersects the axis $\{O : t\}$ with the equation:

$$x = 0 \tag{40}$$

at point 1 with coordinates $(t_1,\, x_1)$

$$-vt_2 = ct_1 - ct_2 \tag{41}$$

$$(c-v)\, t_2 = ct_1 \tag{42}$$

$$t_1 = \left(1 - \frac{v}{c}\right) t_2, \quad x_1 = 0 \tag{43}$$

The straight line

$$x - x_2 = -c\,(t - t_2) \tag{44}$$

.

This straight line intersects the axis

$$t_3 = \left(1 + \frac{v}{c}\right) t_2, \quad x_3 = 0 \tag{45}$$

And now we can calculate t_2':

$$t_2' = \sqrt{t_1 t_3} = \sqrt{\left(1 - \frac{v}{c}\right) t_2 \left(1 + \frac{v}{c}\right) t_2} = t_2 \sqrt{1 - \frac{v^2}{c^2}} \tag{46}$$

To obtain a relationship between t_2', t_2, and x_2, we square the last equation and replace v with $\frac{x_2}{t_2}$.

$$c^2 t_2'^2 = c^2 t_2^2 - x_2^2 \tag{47}$$

But this is the Minkowskian length of the vector $ct_2\vec{e} + x_2\vec{i}$, in the figure vector $|O:2\rangle$. One can show also in the case of a curved world line that a light pulse clock measures the Minkowskian length of its world line. We note that if the light pulse clock were moving at a speed greater than the speed of light c with respect to any reference frame, the light pulse clock would not work. If the light pulse is reflected by one of the mirrors, it will not reach the other mirror, because it will run away with greater speed. If greater speed than that of the light is possible experimentally, then one must reformulate the theory of relativity. But with the method shown here, this would not be too bad. Back to the animation 2., one can also show that besides the relation

$$|O:2|^2 = |O:1| \cdot |O:3| \tag{48}$$

also applies

$$|O:3|^2 = |O:2| \cdot |O:4| \tag{49}$$

This completes the equality of the two reference frames. Here all lengths are Minkowskian lengths. In the 4-dimensional space-time diagram, the light-like vectors play a special role. All of them lie on a 3-dimensional hypersurface with the equation:

$$c^2t^2 - x^2 - y^2 - z^2 = 0 \tag{50}$$

.

If $\vec{x} = ct\hat{\vec{e}} + x\vec{i} + y\hat{\vec{j}} + z\hat{\vec{k}}$ is the 4-dimensional location vector, this equation takes this form:

$$\vec{x}^2 = 0 \tag{51}$$

This is the equation of a hypercone, called a light cone. All vectors that are inside the light cone are time-like, those from outside are space-like. If any vector is time-like/light-like/space-like, then all vectors that are on the same straight line are also time-like/light-like/space-like. Here is the proof: if two vectors are on the same straight line, then there exists a scalar λ such that between the two vectors there exists the relation:

$$\vec{x}_1 = \lambda \vec{x}_2 \tag{52}$$

and therefore:

$$\|\vec{x}_1\| = \lambda^2 \|\vec{x}_2\| \tag{53}$$

The two norms have the same sign (q.e.d.). Therefore the whole world line is called time-like/light-like/space-like. A working clock has a time-like world line. But what condition must two events satisfy to be simultaneous with respect to any clock with a straight world line? As we defined the time of an event above, we draw in the space-time diagram the light cone of event 0. This light cone has the equation:

$$(\vec{x} - \vec{x}_0)^2 = 0 \tag{54}$$

The world line of the clock has the equation:

$$\vec{x} = \vec{a} + \lambda \vec{w} \tag{55}$$

where $\vec{a}$ is the 4-dimensional location vector of a fixed but arbitrarily chosen point of the straight line, and $\vec{w}$ is a constant but otherwise arbitrary vector along the straight line. λ is the parameter of the straight line. To further determine the meaning of the vector $\vec{w}$, we derive the last equation after λ:

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}}\lambda = \vec{w} = c\frac{\mathrm{d}t}{\mathrm{d}}\lambda\left(\vec{e} + \frac{1}{c}\frac{\mathrm{d}x}{\mathrm{d}t}\vec{i} + \frac{1}{c}\frac{\mathrm{d}y}{\mathrm{d}t}\vec{j} + \frac{1}{c}\frac{\mathrm{d}z}{\mathrm{d}t}\vec{k}\right) = c\frac{\mathrm{d}t}{\mathrm{d}\lambda}\left(\vec{e} + \frac{v_x}{c}\vec{i} + \frac{v_y}{c}\vec{j} + \frac{v_z}{c}\vec{k}\right) \tag{56}$$

We choose $\lambda = ct$ such that the coefficient before the bracket vanishes, and the vector $\vec{w}$ becomes:

$$\vec{w} = \vec{e} + \frac{v_x}{c}\vec{i} + \frac{v_y}{c}\vec{j} + \frac{v_z}{c}\vec{k} \tag{57}$$

The clock's world line intersects the light cone in two points 1 and 2, corresponding to parameters λ_1 and λ_2. The middle of the line [1:2] corresponds to the time of the event 0.

$$(\vec{a} + \lambda\vec{w} - \vec{x}_0)^2 = 0 \tag{58}$$

$$\vec{w}^2\lambda^2 + 2\vec{w}\left(\vec{a} - \vec{x}_0\right)\lambda + \left(\vec{a} - \vec{x}_0\right)^2 = 0 \tag{59}$$

This is a second degree equation in λ. Its roots λ_1 and λ_2 are the parameters of the intersections. The parameter of the center is:

$$\lambda_0 = \frac{\lambda_1 + \lambda_2}{2} = \vec{w}\left(\vec{a} - \vec{x}_0\right) \tag{60}$$

So, the condition that two events x_1 and x_2 are simultaneously opposite the clock with velocity $\vec{v}$ is:

$$\vec{w}\left(\vec{a} - \vec{x}_1\right) = \vec{w}\left(\vec{a} - \vec{x}_2\right) \quad \Leftrightarrow \quad \vec{w}\left(\vec{a} - \vec{x}_1\right) - \vec{w}\left(\vec{a} - \vec{x}_2\right) = 0 \tag{61}$$

$$\vec{w}\left(\vec{a} - \vec{x}_1 - \vec{a} + \vec{x}_2\right) = 0 \quad \Leftrightarrow \quad \vec{w}\left(\vec{x}_2 - \vec{x}_1\right) = 0 \quad \Leftrightarrow \quad \vec{w}\left(\vec{x}_1 - \vec{x}_2\right) = 0 \tag{62}$$

The final result of the condition is:

$$c\left(t_1 - t_2\right) - \frac{v_x}{c}\left(x_1 - x_2\right) - \frac{v_y}{c}\left(y_1 - y_2\right) - \frac{v_z}{c}\left(z_1 - z_2\right) = 0 \tag{63}$$

.

Consider again animation 2. The point 6 is the projection in the reference frame S of the point 2 on the {O:t} axis. The point 7 is the projection in the reference frame S' of the point 3 onto the {O:t'

$$\frac{|O:6|}{|O:2|}=\frac{|O:7|}{|O:3|}, \quad |O:6|=t_2, \quad |O:2|=t_2', \quad |O:7|=t_3', \quad |O:3|=t_3' \tag{64}$$

$$\frac{t_2}{t_2'}=\frac{t_3'}{t_3} \tag{65}$$

$$t_3'=\frac{t_2t_3}{t_2'}=\frac{t_2t_3}{t_2\sqrt{1-\frac{v^2}{c^2}}}=\frac{t_3}{\sqrt{1-\frac{v^2}{c^2}}} \tag{66}$$

For point 4, a similar relationship holds as for point 2:

$$t_4'=t_4\sqrt{1-\frac{v^2}{c^2}} \tag{67}$$

But t_4 we can calculate from analytic geometry by the method shown above:

$$\begin{array}{l} x=c\,(t-t_3) \\ \{x=vt \end{array}, \quad x=x_4, \quad t=t_4 \tag{68}$$

$$vt_4=ct_4-ct_3 \tag{69}$$

$$t_4=\frac{t_3}{1-\frac{v}{c}} \tag{70}$$

$$t_4'=\frac{t_3}{1-\frac{v}{c}}\sqrt{1-\frac{v^2}{c^2}}=t_3\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \tag{71}$$

So we want to prove that

$$t_3'\equiv|O:7|=\frac{|O:2|+|O:4|}{2}\equiv\frac{t_2'+t_4'}{2}. \tag{72}$$

$$\frac{t_2'+t_4'}{2}=\frac{1}{2}\left(\frac{t_3}{1+\frac{v}{c}}\sqrt{1-\frac{v^2}{c^2}}+t_3\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}\right)=t_3\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}\cdot\frac{1}{1+\frac{v}{c}}=\frac{t_3}{\sqrt{1-\frac{v^2}{c^2}}}=t_3' \tag{73}$$

q.e.d.

We have shown so far that in the 2-dimensional space-time diagram the direction of the one straight line consisting of simultaneous events with respect to a clock is orthogonal to the clock's world line. We now want to show that this is the direction of the x'-axis. So all straight lines consisting of simultaneous events are parallel to each other. For this we have to prove that the slope of this straight line does not depend on the events it contains, but at most on the velocity v. In animation 2, the straight line consisting of simultaneous events with event 3 opposite observer B', is the straight line

$$m = \frac{x_7 - x_3}{t_7 - t_3} = \frac{x_2 + x_4}{2} - 0\frac{t_2 + t_4}{2} - t_3 = \frac{v\,(t_2 + t_4)}{t_2 + t_4 - 2t_3} = \frac{v}{1 - \frac{2t_3}{t_2 + t_4}} \tag{74}$$

$$t_2 = \frac{t_3}{1 + \frac{v}{c}}, \quad t_4 = \frac{t_3}{1 - \frac{v}{c}} \tag{75}$$

$$\frac{2t_3}{t_2 + t_4} = \frac{2t_3}{\frac{t_3}{1+\frac{v}{c}} + \frac{t_3}{1-\frac{v}{c}}} = \frac{2}{\frac{2}{1-\frac{v^2}{c^2}}} = 1 - \frac{v^2}{c^2} \tag{76}$$

$$m = \frac{v}{1 - \left(1 - \frac{v^2}{c^2}\right)} = \frac{c^2}{v} \tag{77}$$

q.e.d. Thus, the equation of the x'-axis is:

$$x = \frac{c^2}{v}t \tag{78}$$

.

It is parallel with the straight line $\{3:7\}$. And now we want to define the length measurement. We need to compare the length units without stopping the length unit bars. In Figure 7, the straight line $\{A : B'\}$ is the world line of the second end of the scale from the S coordinate system. The distance [O:A] is the unit of length along the axis $\{O : x\}$. The distance [O:A'] is the unit length along the axis $\{O : x'\}$, and the straight line $\{A' : B\}$ is the world line of the end A' of this scale.

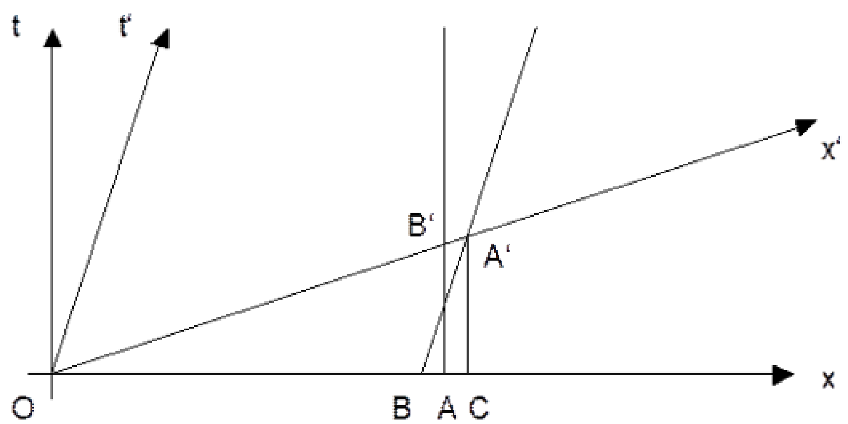


Figure 7:

The distance [O:B'] is the appearance of the scale from observer B to observer B'. The distance [O:B] is the appearance of scale from B' for B. Both observers assert the same proposition, for example: "The appearance of your scale for me is 0.8 of my scale". On the basis of this sentence (including the numerical value) the reference frames cannot be distinguished. Therefore the reference systems are completely equal. And therefore they have equal units of length. Therefore, the Minkowskian length $|O:A|$ is equal to the Minkowskian length $|O:A'|$.

$$\frac{|O:B|}{|O:A|} = \frac{|O:B'|}{|O:A'|} \tag{79}$$

If the point C is the projection of A' onto the axis

$$\frac{|O:B'|}{|O:A'|} = \frac{|O:A|}{|O:C|} \tag{80}$$

so:

$$\frac{|O:B|}{|O:A|} = \frac{|O:A|}{|O:C|}, \quad |O:A'| = |O:A| \tag{81}$$

With the labels:

$$|O:B| = x_B, \quad |O:A| = x_A, \quad |O:C| = x_{A'}, \quad |O:A'| = x'_{A'} \tag{82}$$

thus holds:

$$x_A{}^2 = x_B x_{A'} \tag{83}$$

The point B is the intersection between

$$\begin{cases} x - x_{A'} = v\,(t - t_{A'}) \\ t = 0 \end{cases}$$

$$x_B - x_{A'} = -v \cdot \frac{v}{c^2} x_{A'} = -\frac{v^2}{c^2} x_{A'} \tag{85}$$

$$x_B = \left(1 - \frac{v^2}{c^2}\right) x_{A'} \tag{86}$$

$$x_A{}^2 = \left(1 - \frac{v^2}{c^2}\right) x_{A'}{}^2 = x'_{A'}{}^2 \tag{87}$$

If we replace v in the last equation by $c^2 \frac{t_{A'}}{x_{A'}}$, thus:

$$\frac{v^2}{c^2} = \frac{c^2 {t_{A'}}^2}{{x_{A'}}^2} \tag{88}$$

$$ {x'_{A'}}^2 = {x_{A'}}^2 - c^2 {t_{A'}}^2 \tag{89}$$

If we multiply by an arbitrary λ^2 and then root, we get the Minkowskie length of a space-like vector:

$$s = \sqrt{x^2 - c^2t^2} \tag{90}$$

In summary, the length of an arbitrary vector in the 2-dimensional space-time diagram is $\vec{x}$:

$$s^2 = \sqrt{|\vec{x}^2|} = \sqrt{|c^2t^2 - x^2|} \tag{91}$$

And now we geometrically derive the Lorentz transformation.

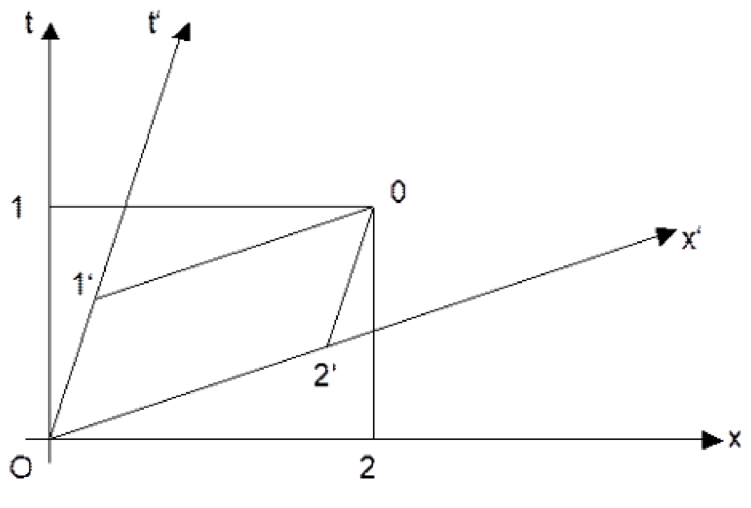


Figure 8:

We use analytic geometry again. Point (1')

$$\begin{cases} x - x_0 = \dfrac{c^2}{v}(t - t_0) & (92) \\ x = vt & (93) \\ , \quad x = x_{1'}, \quad t = t_{1'} & (94) \end{cases}$$

$$t_{1'} - t_0 = \frac{v}{c^2}(x_{1'} - x_0) = \frac{v}{c^2} \cdot vt_{1'} - \frac{v}{c^2}x_0 = \frac{v^2}{c^2}t_{1'} - \frac{v}{c^2}x_0 \tag{95}$$

$$\left(1-\frac{v^2}{c^2}\right)t_{1'}=t_0-\frac{v}{c^2}x_0 \tag{96}$$

$$t_{1'}=\frac{t_0-\frac{v}{c^2}x_0}{1-\frac{v^2}{c^2}},\quad x_{1'}=vt_{1'} \tag{97}$$

The coordinate t' is the length |O:1'| provided with the sign of $t_{1'}$.

$$|ct'|=\sqrt{c^2{t_{1'}}^2-{x_{1'}}^2}=\sqrt{c^2{t_{1'}}^2-v^2{t_{1'}}^2}=|ct_{1'}|\cdot\sqrt{1-\frac{v^2}{c^2}} \tag{98}$$

$$t'=t_{1'}\sqrt{1-\frac{v^2}{c^2}} \tag{99}$$

$$t'=\frac{t_0-\frac{v}{c^2}x_0}{1-\frac{v^2}{c^2}}\cdot\sqrt{1-\frac{v^2}{c^2}} \tag{100}$$

$$t'=\frac{t_0-\frac{v}{c^2}x_0}{\sqrt{1-\frac{v^2}{c^2}}} \tag{101}$$

Point (2')

$$\begin{cases} x-x_0=v\left(t-t_0\right) \\ t=\frac{v}{c^2}x \end{cases},\quad t=t_{2'},\quad x=x_{2'} \tag{102}$$

$$x_{2'}-x_0=v\cdot\frac{v}{c^2}x_{2'}-vt_0=\frac{v^2}{c^2}x_{2'}-vt_0 \tag{103}$$

$$\left(1-\frac{v^2}{c^2}\right)x_{2'}=x_0-vt_0 \tag{104}$$

$$x_{2'}=\frac{x_0-vt_0}{1-\frac{v^2}{c^2}},\quad t_{2'}=\frac{v}{c^2}x_{2'} \tag{105}$$

The coordinate x' is the length |O:2'| provided with the sign of $x_{2'}$. .

$$|x'|=\sqrt{{x_{2'}}^2-c^2{t_{2'}}^2}=\sqrt{{x_{2'}}^2-c^2\cdot\left(\frac{v}{c^2}x_{2'}\right)^2}=\sqrt{{x_{2'}}^2-\frac{v^2}{c^2}{x_{2'}}^2}=|x_{2'}|\cdot\sqrt{1-\frac{v^2}{c^2}} \tag{106}$$

$$x'=x_{2'}\sqrt{1-\frac{v^2}{c^2}} \tag{107}$$

$$x' = \frac{x_0 - vt_0}{1 - \frac{v^2}{c^2}} \cdot \sqrt{1 - \frac{v^2}{c^2}} \tag{108}$$

$$x' = \frac{x_0 - vt_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{109}$$

With the labels:

$$x = x_0, \quad t = t_0, \tag{110}$$

is the coordinate transformation:

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{111}$$

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{112}$$

This is the well-known Lorentz transformation for the coordinates t and x.