Euclidean coordinates in relativity spacetime diagram

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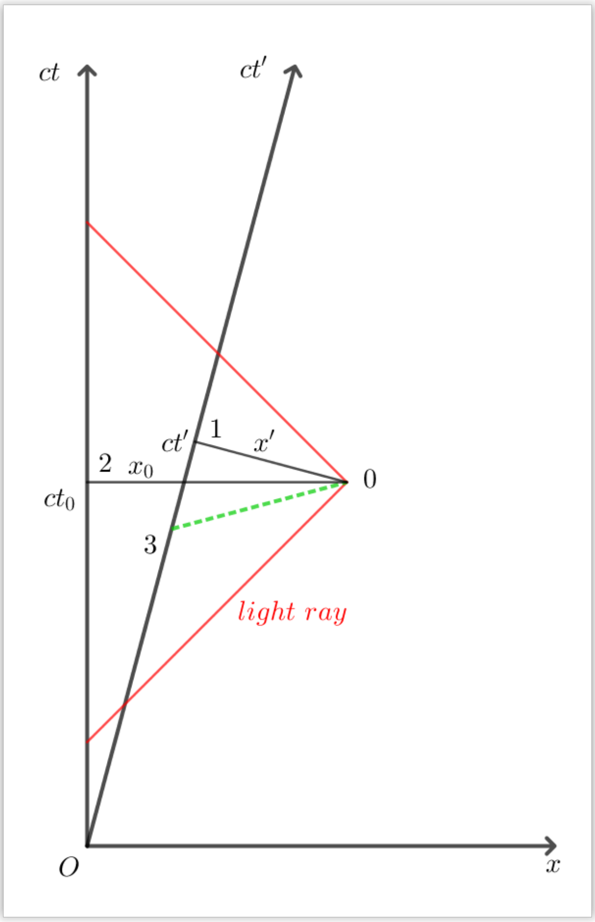
1 Mathematical introduction

The physical situation consists of two observers S and S' which are in rectilinear and uniform translational motion relative to each other. The x and x' axes coincide and the motion is along them in 3-D. Each observer has a clock located at the origin of the 3-D axes. The observers emit electromagnetic TV signals containing the image of their own clock and re-emit the received image of the other.
Any event is a reflection of this signal. We write the equations of the lines in the space-time diagram, and the intersection of two lines is an event.



$$\begin{cases} c = 1 & (1) \\ t' = at + fx & (2) \\ x' = a(x - vt) & (3) \\ a^2 + avf = 1 & (4) \end{cases}$$

Galilei:

$$1^2 + 1 \cdot v \cdot 0 = 1 \tag{5}$$

$$\begin{cases} t' = t & (6) \\ x' = x - vt & (7) \end{cases}$$

Euclid:

$$\cos^2\varphi + \underbrace{\cos\varphi}_{a}\underbrace{\frac{\sin\varphi}{\cos\varphi}}_{v}\underbrace{\sin\varphi}_{f} = 1 \tag{8}$$

$$\begin{cases} t' = t\cos\varphi + x\sin\varphi & (9) \\ x' = -t\sin\varphi + x\cos\varphi & (10) \end{cases}$$

$$s^2 = t^2 + x^2 = t'^2 + x'^2 \tag{11}$$

$$\begin{cases} t' = s\cos\theta' & (12) \\ x' = s\sin\theta' & (13) \\ t = s\cos\theta & (14) \\ x = s\sin\theta & (15) \end{cases}$$

$$\begin{cases} \cos\theta' = \cos\theta\cos\varphi + \sin\theta\sin\varphi = \cos(\theta - \varphi) & (16) \\ \sin\theta' = -\cos\theta\sin\varphi + \sin\theta\cos\varphi = \sin(\theta - \varphi) & (17) \end{cases}$$

$$\boxed{\theta' = \theta - \varphi} \tag{18}$$

Lorentz:

$$\cosh^2\psi + \underbrace{\cosh\psi}_{a}\underbrace{\frac{\sinh\psi}{\cosh\psi}}_{v}\underbrace{(-\sinh\psi)}_{f} = 1 \tag{19}$$

$$\begin{cases} t' = t\cosh\psi - x\sinh\psi & (20) \\ x' = -t\sinh\psi + x\cosh\psi & (21) \end{cases}$$

$$\tau^2 = t^2 - x^2 = t'^2 - x'^2 \tag{22}$$

$$\begin{cases} t' = \tau \cosh \alpha' & (23) \\ x' = \tau \sinh \alpha' & (24) \\ t = \tau \cosh \alpha & (25) \\ x = \tau \sinh \alpha & (26) \end{cases}$$

$$\begin{cases} \cosh \alpha' = \cosh \alpha \cosh \psi - \sinh \alpha \sinh \psi = \cosh(\alpha - \psi) & (27) \\ \sinh \alpha' = -\cosh \alpha \sinh \psi + \sinh \alpha \cosh \psi = \sinh(\alpha - \psi) & (28) \end{cases}$$

$$\boxed{\alpha' = \alpha - \psi} \tag{29}$$

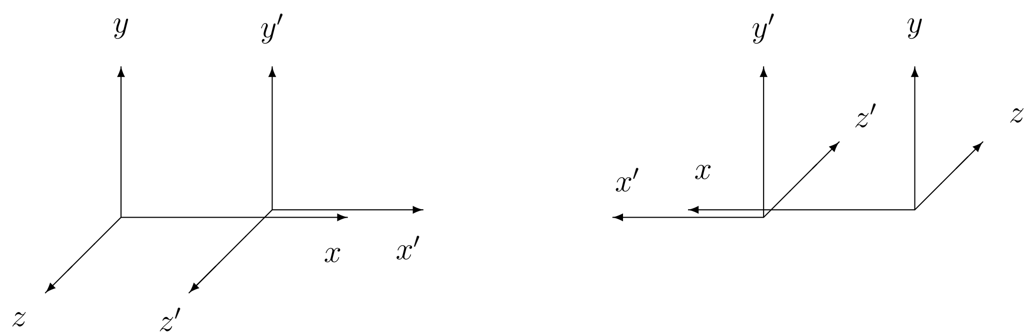


Figure 1:

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{30}$$

$$\Lambda = \begin{pmatrix} a_0 & b_0 & c_0 & d_0 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix} \tag{31}$$

$$R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{32}$$

$$X' = \Lambda X \tag{33}$$

$$X' \to R_2 X', \quad X \to R_2 X \tag{34}$$

$$R_2\Lambda = \Lambda R_2 \tag{35}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_0 & b_0 & c_0 & d_0 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 & c_0 & d_0 \\ a_1 & b_1 & c_1 & d_1 \\ -a_3 & -b_3 & -c_3 & -d_3 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix} \tag{36}$$

$$\begin{pmatrix} a_0 & b_0 & c_0 & d_0 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 & -d_0 & c_0 \\ a_1 & b_1 & -d_1 & c_1 \\ a_2 & b_2 & -d_2 & c_2 \\ a_3 & b_3 & -d_3 & c_3 \end{pmatrix} \tag{37}$$

$$\Lambda = \begin{pmatrix} a_0 & b_0 & 0 & 0 \\ a_1 & b_1 & 0 & 0 \\ 0 & 0 & c_3 & c_2 \\ 0 & 0 & -c_2 & c_3 \end{pmatrix} \tag{38}$$

$$R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{39}$$

$$X' \to R_3 X', \quad X \to R_3 X \tag{40}$$

$$R_3\Lambda = \Lambda R_3 \tag{41}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a_0 & b_0 & 0 & 0 \\ a_1 & b_1 & 0 & 0 \\ 0 & 0 & c_3 & c_2 \\ 0 & 0 & -c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 & 0 & 0 \\ a_1 & b_1 & 0 & 0 \\ 0 & 0 & -c_2 & c_3 \\ 0 & 0 & -c_3 & -c_2 \end{pmatrix} \tag{42}$$

$$\begin{pmatrix} a_0 & b_0 & 0 & 0 \\ a_1 & b_1 & 0 & 0 \\ 0 & 0 & c_3 & c_2 \\ 0 & 0 & -c_2 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 & 0 & 0 \\ a_1 & b_1 & 0 & 0 \\ 0 & 0 & c_2 & c_3 \\ 0 & 0 & -c_3 & c_2 \end{pmatrix} \tag{43}$$

$$\Lambda = \begin{pmatrix} a_0 & b_0 & 0 & 0 \\ a_1 & b_1 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_3 \end{pmatrix} \tag{44}$$

$$X' \to R_1 X, \quad X \to R_1 X' \tag{45}$$

$$R_1 X = \Lambda R_1 X' = \Lambda R_1 \Lambda X, \quad R_1^2 = \mathbb{1} \tag{46}$$

$$\Lambda R_1 = \begin{pmatrix} a_0 & b_0 & 0 & 0 \\ a_1 & b_1 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 & 0 & 0 \\ a_1 & -b_1 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & -c_3 \end{pmatrix} \tag{47}$$

$$(\Lambda R_1)^2 = \begin{pmatrix} a_0 & b_0 & 0 & 0 \\ a_1 & -b_1 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & -c_3 \end{pmatrix} \begin{pmatrix} a_0 & b_0 & 0 & 0 \\ a_1 & -b_1 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & -c_3 \end{pmatrix} = \tag{48}$$

$$= \begin{pmatrix} a_0^2 - a_1 b_0 & a_0 b_0 - b_0 b_1 & 0 & 0 \\ a_0 a_1 - a_1 b_1 & a_1 b_0 + b_1^2 & 0 & 0 \\ 0 & 0 & c_3^2 & 0 \\ 0 & 0 & 0 & c_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{49}$$

$$\Lambda = \begin{pmatrix} a_0 & -va_0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{50}$$

$$\boxed{a_0^2 + va_0 a_1 = 1}, \quad c = 1 \tag{51}$$

Galilei:

$$1^2 + 1 \cdot v \cdot 0 = 1 \tag{52}$$

$$\begin{cases} t' = t & (53) \\ x' = x - vt & (54) \end{cases}$$

Euclid:

$$\cos^2\varphi + \underbrace{\cos\varphi}_{a_0} \underbrace{\frac{\sin\varphi}{\cos\varphi}}_{v} \underbrace{\sin\varphi}_{a_1} = 1 \tag{55}$$

$$\begin{cases} t' = t\cos\varphi + x\sin\varphi & (56) \\ x' = -t\sin\varphi + x\cos\varphi & (57) \end{cases}$$

$$s^2 = t^2 + x^2 = t'^2 + x'^2 \tag{58}$$

$$\begin{cases} t' = s\cos\theta' & (59) \\ x' = s\sin\theta' & (60) \\ t = s\cos\theta & (61) \\ x = s\sin\theta & (62) \end{cases}$$

$$\begin{cases} \cos\theta' = \cos\theta\cos\varphi + \sin\theta\sin\varphi = \cos(\theta-\varphi) & (63) \\ \sin\theta' = -\cos\theta\sin\varphi + \sin\theta\cos\varphi = \sin(\theta-\varphi) & (64) \end{cases}$$

$$\boxed{\theta' = \theta - \varphi} \tag{65}$$

Lorentz:

$$\cosh^2\psi + \underbrace{\cosh\psi}_{a_0}\underbrace{\frac{\sinh\psi}{\cosh\psi}}_{v}\underbrace{(-\sinh\psi)}_{a_1} = 1 \tag{66}$$

$$\begin{cases} t' = t\cosh\psi - x\sinh\psi & (67) \\ x' = -t\sinh\psi + x\cosh\psi & (68) \end{cases}$$

$$\tau^2 = t^2 - x^2 = t'^2 - x'^2 \tag{69}$$

$$\begin{cases} t' = \tau\cosh\alpha' & (70) \\ x' = \tau\sinh\alpha' & (71) \\ t = \tau\cosh\alpha & (72) \\ x = \tau\sinh\alpha & (73) \end{cases}$$

$$\begin{cases} \cosh\alpha' = \cosh\alpha\cosh\psi - \sinh\alpha\sinh\psi = \cosh(\alpha-\psi) & (74) \\ \sinh\alpha' = -\cosh\alpha\sinh\psi + \sinh\alpha\cosh\psi = \sinh(\alpha-\psi) & (75) \end{cases}$$

$$\boxed{\alpha' = \alpha - \psi} \tag{76}$$

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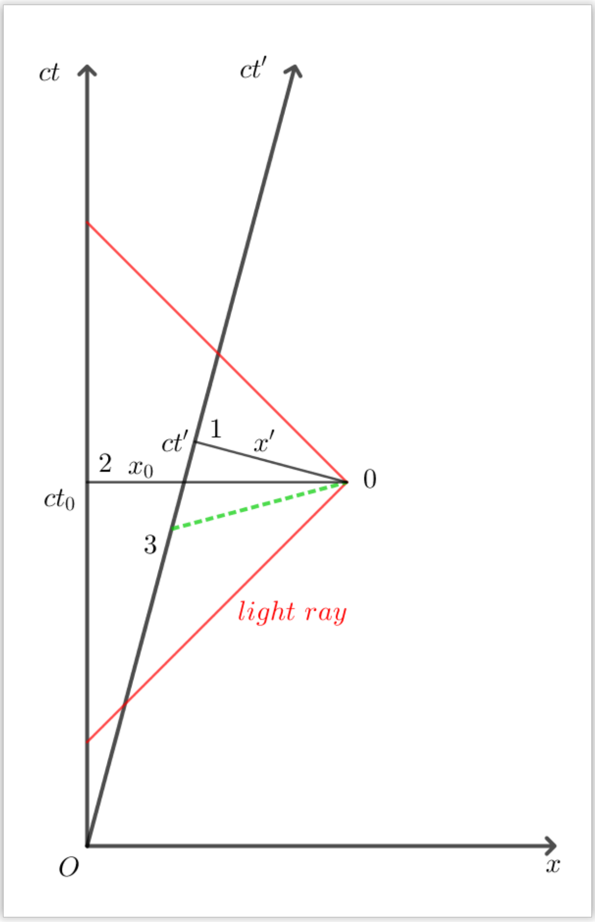
2 Euclidean spacetime



$$\begin{cases} x - x_0 = -\dfrac{c^2}{v}(t - t_0) & (77) \\ x = vt & (78) \\ t = t_1 & (79) \\ x = x_1 & (80) \end{cases}$$

$$vt_1 - x_0 = -\frac{c^2}{v}t_1 + \frac{c^2}{v}t_0 \tag{81}$$

$$\left(v + \frac{c^2}{v}\right) t_1 = x_0 + \frac{c^2}{v}t_0 \tag{82}$$

$$\left(\frac{v^2}{c^2} + 1\right) t_1 = \frac{v}{c^2}x_0 + t_0 \tag{83}$$

$$\begin{cases} t_1 = \dfrac{t_0 + \dfrac{v}{c^2}x_0}{1 + \dfrac{v^2}{c^2}} & (84) \\ x_1 = vt_1 & (85) \end{cases}$$

$$c^2 t'^2 = c^2 t_1^2 + x_1^2 = c^2 t_1^2 + v^2 t_1^2 = (c^2 + v^2) t_1^2 = \left(1 + \frac{v^2}{c^2}\right) c^2 t_1^2 \tag{86}$$

$$\boxed{t' = \frac{t_0 + \frac{v}{c^2} x_0}{\sqrt{1 + \frac{v^2}{c^2}}}} \tag{87}$$

$$t_1 - t_0 = \frac{t_0 + \frac{v}{c^2} x_0 - t_0 - \frac{v^2}{c^2} t_0}{1 + \frac{v^2}{c^2}} = \frac{\frac{v}{c^2}(x_0 - v t_0)}{1 + \frac{v^2}{c^2}} \tag{88}$$

$$x_0 - x_1 = x_0 - v t_1 = x_0 - \frac{v t_0 + \frac{v^2}{c^2} x_0}{1 + \frac{v^2}{c^2}} = \frac{x_0 + \frac{v^2}{c^2} x_0 - v t_0 - \frac{v^2}{c^2} x_0}{1 + \frac{v^2}{c^2}} = \frac{x_0 - v t_0}{1 + \frac{v^2}{c^2}} \tag{89}$$

$$x'^2 = [c(t_1 - t_0)]^2 + (x_0 - x_1)^2 = \left(1 + \frac{v^2}{c^2}\right) \left(\frac{x_0 - v t_0}{1 + \frac{v^2}{c^2}}\right)^2 \tag{90}$$

$$\boxed{x' = \frac{x_0 - v t_0}{\sqrt{1 + \frac{v^2}{c^2}}}} \tag{91}$$

$$\begin{cases} t' = \dfrac{t_0 + \frac{v}{c^2} x_0}{\sqrt{1 + \frac{v^2}{c^2}}} & (92) \\[2ex] x' = \dfrac{x_0 - v t_0}{\sqrt{1 + \frac{v^2}{c^2}}} & (93) \end{cases}$$