Proof of the constancy of the speed of light

Ladislau Radu

1 Derivation of Lorentz transformation equations

.
The equations must be linear:

$$x' = ax + bt \tag{1}$$

$$t' = fx + gt \tag{2}$$

We want to calculate the expressions of the coefficients a, b, f, g....
For this we postulate the homogeneity and isotropy of space, and from this we prove the existence of a universal constant with the size of a velocity, which is identified with the speed of light in vacuum.
We consider the motion of the two origins of coordinate systems.

$$x = 0 \text{ implies } x' = v't' \text{ and } x' = 0 \text{ implies } x = vt \tag{3}$$

$$0 = avt + bt \quad b = -av \tag{4}$$

Thus, equation (1) becomes:

$$x' = a(x - vt) \tag{5}$$

From Figure 1 we can see that the following transformation leaves the whole situation invariant...

$$\begin{array}{lll} x' & \rightarrow & -x \\ t' & \rightarrow & t \end{array} \qquad \begin{array}{lll} x & \rightarrow & -x' \\ t & \rightarrow & t' \end{array} \tag{6}$$

We replace conditions (6) in equation (5)

$$-x = a(-x' - vt') \quad \Longrightarrow \quad x = a(x' + vt') \tag{7}$$

$$(7)\&(3) \;\; => \;\; 0 = a(v't' + vt') \;\; => \;\; \boxed{v' = -v} \tag{8}$$

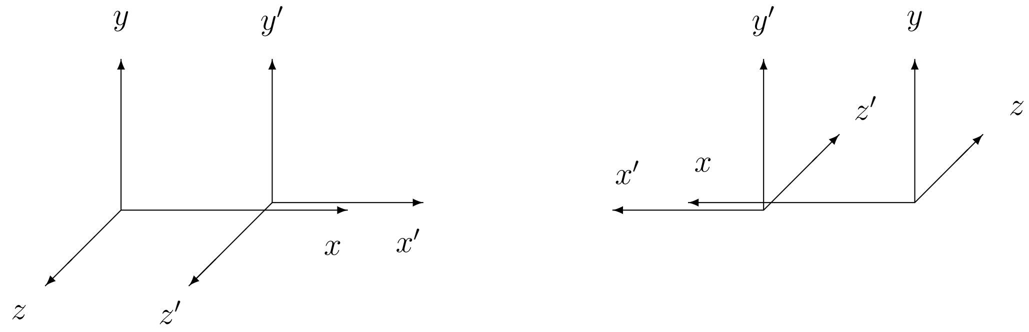

Figure 1:

$$x = a(a(x - vt) + v(fx + gt)) = (a^2 + avf)x + (-a^2 v + avg)t$$

$$= a(a + vf)x + av(-a + g)t \tag{9}$$

$$g = a \quad a(a + vf) = 1 \tag{10}$$

The coefficients a and f we replace with two other constants γ and h by substitution:

$$a = \gamma; \quad f = -\gamma\frac{v}{h} \tag{11}$$

In this way, the transformation equations take the form:

$$x' = \gamma(x - vt) \tag{12}$$

$$t' = \gamma\left(t - \frac{v}{h}x\right) \tag{13}$$

We account for (10) in equation (9):

$$1 = a(a + vf) = \gamma\left(\gamma - \gamma\frac{v^2}{h}\right) = \gamma^2\left(1 - \frac{v^2}{h}\right) \tag{14}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{h}}} \tag{15}$$

$$x' = \gamma(x - vt) \tag{16}$$

$$t' = \gamma\left(t - \frac{v}{h}x\right) \tag{17}$$

A rigorous derivation.

$$\Lambda = \begin{pmatrix} a & b \\ f & g \end{pmatrix} \qquad \Lambda' = \begin{pmatrix} a' & b' \\ f' & g' \end{pmatrix} \qquad X' = \begin{pmatrix} x' \\ t' \end{pmatrix} \qquad X = \begin{pmatrix} x \\ t \end{pmatrix} \tag{18}$$

$$X' = \Lambda X \qquad X = \Lambda' X' \qquad \Lambda'\Lambda = \Lambda\Lambda' = \mathbb{1} \tag{19}$$

$$\begin{pmatrix} a' & b' \\ f' & g' \end{pmatrix} \cdot \begin{pmatrix} a & b \\ f & g \end{pmatrix} = \begin{pmatrix} a'a + b'f & a'b + b'g \\ f'a + g'f & f'b + g'g \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{20}$$

$$t'\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ f & g \end{pmatrix}\begin{pmatrix} v \\ 1 \end{pmatrix} t \quad => \quad \boxed{av + b = 0} \tag{21}$$

$$t\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a' & b' \\ f' & g' \end{pmatrix}\begin{pmatrix} v' \\ 1 \end{pmatrix} t' \quad => \quad \boxed{a'v' + b' = 0} \tag{22}$$

$$R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad R^2 = \mathbb{1} \tag{23}$$

$$X' \to RX, \qquad X \to RX' \tag{24}$$

$$RX = \Lambda R X' = \Lambda R \Lambda X \tag{25}$$

$$(\Lambda R)^2 = \mathbb{1} \tag{26}$$

$$\Lambda R = \begin{pmatrix} a & b \\ f & g \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -a & b \\ -f & g \end{pmatrix} \tag{27}$$

$$(\Lambda R)^2 = \begin{pmatrix} -a & b \\ -f & g \end{pmatrix} \cdot \begin{pmatrix} -a & b \\ -f & g \end{pmatrix} = \begin{pmatrix} a^2 - bf & -ab + bg \\ af - fg & -bf + g^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{28}$$

$$b = -av \tag{29}$$

$$-ab + bg = 0 => \quad a = g \tag{30}$$

$$a^2 - bf = 1 \quad => \quad 1 = a^2 + avf = a(a + vf) \tag{31}$$

$$a = \gamma, \quad f = -\gamma \frac{v}{h} \tag{32}$$

$$1 = a(a + vf) = \gamma \left(\gamma - \gamma \frac{v^2}{h} \right) = \gamma^2 \left(1 - \frac{v^2}{h} \right) \tag{33}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{h}}} \tag{34}$$

$$x' = \gamma(x - vt) \tag{35}$$

$$t' = \gamma \left(t - \frac{v}{h} x \right) \tag{36}$$

2 Demonstration of the constancy of the speed of light

We want to investigate the significance of the constant h. From (14) we see that h is a velocity squared. Imposing the condition that the transformation equations form a group, it follows:
1. The product of two successive transformations is an internal operation.

$$\Lambda(v, h) = \gamma(v, h) \begin{pmatrix} 1 & -v \\ -\frac{v}{h} & 1 \end{pmatrix} \tag{37}$$

$$\Lambda(v_1, h_1)\Lambda(v_2, h_2) = \gamma(v_1, h_1)\gamma(v_2, h_2) \begin{pmatrix} 1 & -v_1 \\ -\frac{v_1}{h_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & -v_2 \\ -\frac{v_2}{h_2} & 1 \end{pmatrix}$$

$$= \gamma_1\gamma_2 \begin{pmatrix} 1 + \frac{v_1 v_2}{h_2} & -v_1 - v_2 \\ -\frac{v_1}{h_1} - \frac{v_2}{h_2} & 1 + \frac{v_1 v_2}{h_1} \end{pmatrix} \tag{38}$$

A necessary condition for the product of linear operators to be an internal operation is that the elements on the first diagonal are equal, so $h_1 = h_2$. Let us show sufficiency.

$$\Lambda_1\Lambda_2 = \frac{1+\dfrac{v_1v_2}{h}}{\sqrt{\left(1-\dfrac{v_1^2}{h}\right)\left(1-\dfrac{v_2^2}{h}\right)}}\begin{pmatrix} 1 & -\dfrac{v_1+v_2}{1+\dfrac{v_1v_2}{h}} \\ -\dfrac{\dfrac{v_1+v_2}{h}}{1+\dfrac{v_1v_2}{h}} & 1 \end{pmatrix} \tag{39}$$

By direct calculation it is shown, that

$$\Lambda(v_1)\Lambda(v_2) = \Lambda\left(\frac{v_1+v_2}{1+\dfrac{v_1v_2}{h}}\right) \tag{40}$$

2. Associativity follows from the associativity of the product of matrices.
3. The neutral element is obtained for $v = 0$, $\Lambda(0) = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
4. The inverse element is:

$$\Lambda(v)^{-1} = \Lambda(-v) \tag{41}$$

It follows that h must be independent of the coordinate transformation, hence of v. Let us consider, then, that the event lies on a universe line such that:

$$x' = u't' \quad x = ut \tag{42}$$

We divide (15) by (16):

$$\frac{x'}{t'} = \frac{\dfrac{x}{t}-v}{1-\dfrac{xv}{ht}} \tag{43}$$

From relation (42) we express on h, obtaining:

$$\frac{1}{h} = \left(\frac{1}{u}-\frac{1}{u'}\right)\frac{1}{v}+\frac{1}{uu'} \qquad \frac{u'u}{h} = \frac{u'-u}{v}+1 \tag{44}$$

But h cannot depend on v, the coefficient in parentheses must cancel. So there must be a velocity w, which is a universal constant, so that:

$$u' = u = w \text{ and } h = u'u = w^2 \tag{45}$$

This velocity has been identified as the speed of light in vacuum “c”.

$$w = c,\ h = c^2 \tag{46}$$

We have thus demonstrated the sufficiency of the existence of an invariant velocity in a Lorentz transformation. It is, at the same time, constant because it is independent of the coordinates. It can only be a universal constant, since it does not depend on the velocity v either.
Necessity is not true because there are, in relation (43), other possibilities. For example

$$\frac{u' - u}{v} = -1, \quad u' = u - v \tag{47}$$

which leads to the Galilei transformation. In this case $h = \infty$. We can by no means rule out the Galilei transformation.
One can prove, however, the uniqueness of the module of the finite solution. If there is a finite velocity c invariant to Lorentz transformations, then it is unique.
Proof.
We denote the invariant velocity by "c". Suppose, that there is still a finite invariant velocity c_1.
Then:..

$$\frac{u'u}{c^2} = \frac{u' - u}{v} + 1 \tag{48}$$

and

$$u' = u = c_1 \tag{49}$$

Substituting (48) into (47), we obtain:

$$\frac{c_1^2}{c^2} = 1 \tag{50}$$

$$|c_1| = |c| \tag{51}$$

Q.E.D.
Conclusion: the speed of light in vacuum c_0 can be expressed as

$$c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \tag{52}$$

with the notations established in electrodynamics. The speed of light in any medium varies in the medium, possibly in both time and space. But the absolute velocity which has the value c_0 remains constant.
On the other hand the absolute speed c_0 is not specific to light. There are many elementary particles that propagate at this speed, namely those with zero rest mass.
Unfortunately, efforts made by non-specialists to contradict the "principle of the constancy of the speed of light" are doomed to failure.

References

[1] Chris Doran and Anthony Lasenby. *Geometric Algebra for Physicists*. Cambridge University Press, 2003.

[2] Steven Weinberg. *Gravitation and Cosmology*. John Wiley & Sons, Inc., 1972.