Derivation of mass-energy relationship $\mathbf{E = mc^2}$

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We start from the second principle of mechanics, written with the help of the momentum

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{1}$$

and we express the kinetic energy W

$$dW = \vec{F}\,d\vec{r} \tag{2}$$

according to the kinetic energy variation theorem. Further, we obtain:
We denote everywhere in this article, the scalar product with point or juxtaposition, and the square of a vector will be the scalar product of the vector with itself

$$-dW\,dt + d\vec{p}\,d\vec{r} = 0 \tag{3}$$

But, we have established in relativistic kinematics the expression

$$cdt\vec{e} \oplus d\vec{r} = d\vec{x} \tag{4}$$

introducing the notions of spacetime, quadravector and Lorentz transformation. c is the speed of light in vacuum, and $\vec{e}$ is the unit vector on the Ot axis, with property $\vec{e}^2 = -1$. The operator $\oplus$ is the direct sum between vector spaces. Since 0 is a scalar and $d\vec{x}$ is a quadravector, the following expression is also a quadravector.

$$d\vec{P} = \frac{dW}{c}\,\vec{e} \oplus d\vec{p} \tag{5}$$

and by integration...

$$\vec{P} = \frac{W}{c}\vec{e} \oplus \vec{p} \tag{6}$$

since relation (3) can be written

$$d\vec{P} \cdot d\vec{x} = \left(\frac{dW}{c}\,\vec{e} \oplus d\vec{p}\right)(cdt\vec{e} \oplus d\vec{r}) = -dW\,dt + d\vec{p} \cdot d\vec{r} = 0$$

The quadrivector $\vec{P}$ is called the energy-momentum vector because its coordinates have the dimension of a momentum, so its square is invariant and constant and has dimension $[(\text{mass}) \times (\text{velocity})]^2$. Multiplying relation (6) by itself, we get:

$$\left(\frac{W}{c}\vec{e} \oplus \vec{p}\right) \cdot \left(\frac{W}{c}\vec{e} \oplus \vec{p}\right) = -\frac{W^2}{c^2} + \vec{p}^2 = \vec{P}^2 \quad => \quad \vec{p}^2 = \frac{W^2}{c^2} + \vec{P}^2 \tag{7}$$

We return to relation (3), where we consider the classical definition of momentum, $\vec{p} = \tilde{m}\vec{v} = \tilde{m}\frac{d\vec{r}}{dt}$. We have:

$$-dW dt + d\vec{p}d\vec{r} = 0 \quad \Rightarrow \quad d\vec{p}\frac{d\vec{r}}{dt} = dW \quad \Rightarrow$$

$$\vec{p}d\vec{p} = \tilde{m}dW \quad \Rightarrow \quad \frac{d(\vec{p}^2)}{dW} = 2\tilde{m} \tag{8}$$

We use in (8) the relation (7)....

$$2m = \frac{d(\vec{p}^2)}{dW} = \frac{d}{dW}\left(\frac{W^2}{c^2} + \vec{P}^2\right) = \frac{2W}{c^2} \tag{9}$$

$$\boxed{W = \tilde{m}c^2} \tag{10}$$

Using (10) in (7)....—

$$-\tilde{m}^2c^2 + \tilde{m}^2v^2 = \vec{P}^2$$

$$\Downarrow$$

$$\tilde{m}^2 = \frac{-\vec{P}^2}{c^2 - \vec{v}^2} \tag{11}$$

Here $\vec{P}^2$ is invariant, so constant (independent of velocity), and $\tilde{m}(\vec{v})$ is the classical mass called the mass of motion, because it depends on velocity. In the eigen-system of the particle (for which $v = 0$), we have: $\tilde{m}(v = 0)^2 = \frac{-\vec{P}^2}{c^2}$. So the constant $\frac{-\vec{P}^2}{c^2}$ is the square of a quantity, which represents the mass of the particle at zero velocity. We denote this quantity as m_0 and call it the rest mass:

$$\frac{-\vec{P}^2}{c^2} = m_0^2$$

$$\tilde{m}^2 = \frac{-\vec{P}^2}{c^2 - v^2} = \frac{m_0^2c^2}{c^2 - v^2}$$

$$\tilde{m} = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{12}$$

Energy conservation in special relativity theory

We assume that the force F comes from a scalar potential independent of time, as in classical mechanics.

$$\vec{F} = -\nabla V \tag{13}$$

We overlook the fact, that the equations describing the physical system are not Lorentz covariant, because they are generally covariant.
Then we have for kinetic energy:

$$E_k = \int_0^v dW = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2 \tag{14}$$

Next we have:

$$dW = \vec{F} \cdot d\vec{r} = -\nabla V \, d\vec{r} = -dV \tag{15}$$

$$d(W + V) = 0 \quad => \quad W + V = const. \tag{16}$$

so the physical quantity $E = W + V$ is conserved.

References

[1] Chris Doran and Anthony Lasenby. *Geometric Algebra for Physicists.* Cambridge University Press, 2003.

[2] Steven Weinberg. *Gravitation and Cosmology.* John Wiley & Sons, Inc., 1972.